Continuous limit of the Nagel-Schreckenberg model

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A generalized version of the Nagel-Schreckenberg model of traffic flow is presented that allows for continuous values of the velocities and spatial coordinates. It is shown that this generalization reveals structures of the dynamics that are masked by the discreteness of the original model and thus helps to clarify the physical interpretation of the dynamics considerably. It is shown numerically that the transition leading from the free flow regime to the congested flow regime bears strong similarities with a first-order phase transition in equilibrium thermodynamics. A similar behavior is observed in more complicated microscopic models and in hydrodynamical descriptions of traffic flow, putting the model within a broader context of other models of traffic flow. An additional advantage of this continuous model is that it is much easier to calibrate with empirical data, only slightly decreasing numerical efficiency. [S1063-651X(96)08810-1]

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I. THE NAGEL-SCHRECKENBERG MODEL

There are many model approaches to describe traffic in a more or less detailed way [1–5]. One approach that stands out for its simplicity is the simulation of traffic using cellular automata [6–11]. The idea behind this approach is that extremely crude microscopic modeling, based on a caricature of individual driver behavior, may be sufficient to capture the main macroscopic aspects of traffic, like the fundamental diagram, appearance of traffic jams, and so on.

The model proposed by Nagel and Schreckenberg in [6] tries to model basically two properties of road traffic: (1) Cars travel at some desired speed, unless they are forced to slow down to avoid collisions with other vehicles. (2) Interactions are short ranged and can be approximated as being restricted to nearest neighbors. In the model approach it is assumed that imperfections in the way drivers react can be modeled as noise.

The velocity and consequently the positions of the cars can only assume integer values between 0 and $v_{\max}$, where $v_{\max}$ itself is an integer. Comparisons with measured data show that $v_{\max}$ should be no larger than 3 [12] to acquire correct values for the density where the model displays its maximum flow.

Clearly the discreteness of the model does not correspond to any property of real traffic. Therefore the question comes up naturally if there are any properties of the cellular automaton dynamics that can be identified as consequences of the discrete structure of the model. To answer this question we will present a model in which the state variables assume real values. This model can be interpreted as the limiting case of a series of cellular automaton models with different resolutions.

II. NAIVE CONTINUOUS LIMIT

Before actually performing the continuous limit we first have to consider the scale transformation between the model and reality to find out how the limit has to be performed. Let us assume that there is a maximum density of cars $\rho_{\max}$ in the system (which was implicitly assumed to be equal to 1 in the original model). Setting $\rho_{\max}$ to values smaller than 1 corresponds to a finer spatial resolution. Obviously the continuous limit is performed letting $v_{\max}\to\infty$ and $\rho_{\max}\to 0$. Since the product $v_{\max}\rho_{\max}$ determines the time scale to which one time step in the model corresponds, the continuous limit has to be performed in such a way that $v_{\max}\rho_{\max}$ is kept constant.

In the cellular automaton (CA) noise is introduced by decelerating cars by an amount of 1 (which is equal to the maximum acceleration) randomly with probability p_{brake} (usually set to 1/2). In the models with higher spatial resolution (where $v_{\max}$ and consequently the maximum acceleration $a_{\max}$ go to infinity) this is generalized to an equipartition between zero and $a_{\max}$. Although this generalization appears to be straightforward, we will see that it does not yield an optimum agreement between successive models in the limiting process. The cellular automaton rules for the intermediate models are given in the Appendix.

Figure 1 shows the fundamental diagrams for a succession of models and the continuous limit. The update rules used for the continuous model are given as follows:

$$v_{\mathrm{des}}=\min[v(t)+a_{\max},v_{\max},s_{\mathrm{gap}}(t)],$$
$$v(t+1)=\max[0,v_{\mathrm{des}}-\sigma n_{\mathrm{ran},0,1}], \tag{1}$$
$$x(t+1)=x(t)+v(t+1),$$

where $s_{\mathrm{gap}}(t)$ is the free space to the car ahead, $a_{\max}$ is the maximum acceleration, $n_{\mathrm{ran},0,1}$ is a random number in the interval (0,1), and σ is the maximum deceleration due to noise. For the continuous model that is reached as the limit of the sequence of CA models the parameters are $a_{\max}=\sigma=1$, but they can, of course, have different values.

There are two features visible in Fig. 1 that arise from the changes. The first one is that the system appears to become less sensitive against the noise, leading to a significantly increased flow. However, this increase is due to a difference in the statistics of the noise in the discrete and the continuous model, as will be shown below. Second, the behavior of the

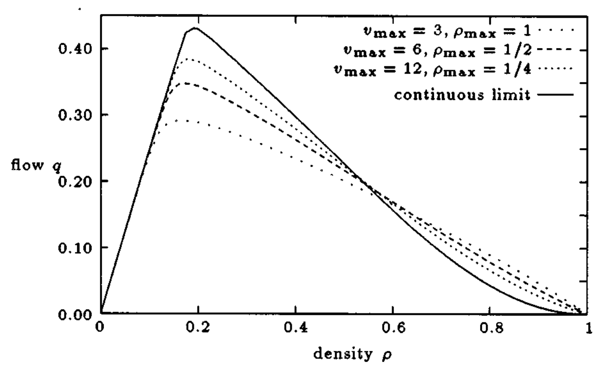

FIG. 1. Sequence of models converging towards the continuous limit.

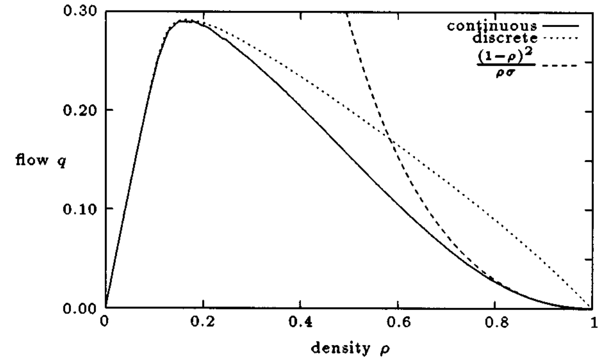

FIG. 2. Comparison between the fundamental diagram of the discrete and the continuous version of the Nagel-Schreckenberg model. The parameters chosen for the continuous model correspond to the $p_{brake}=0.5$, $v_{max}=3$ case of the discrete model. We have used $a_{max}=(1+\sqrt{3})/2$, $\sigma=\sqrt{3}$ and $v_{max}=2.56+\sqrt{3}/2$. The asymptotic behavior is drawn from Eq. (3).

fundamental diagram changes for very high densities in such a way that the curve approaches the value zero with vanishing slope.

III. INTERPRETATION OF THE FEATURES

We will only give qualitative explanations for the effects and limit our attention to the continuous model.

The main difference between the continuous model and the CA at high densities is that in the continuous case gaps of a length smaller than 1 can appear in the system. Loosely speaking those gaps are less likely to be used efficiently by the individual drivers than the larger ones. This is in fact quite realistic and appears to be of great importance when the model is calibrated. For a quantitative analysis note that for very high densities a car having a gap g in front of the bumper will assume a velocity of g before the randomization step. It is then decelerated to some velocity in the interval $(0,g)$. The probability that it is decelerated to a nonzero value is g/σ, where any of these values is assumed with equal probability. So the mean velocity of the car will be

$$\langle v\rangle_{\text{gap}=g}=\frac{g^2}{2\sigma} \tag{2}$$

after the update. Now it is very plausible and confirmed by numerical calculations that the gap distribution becomes strictly exponential for very high densities. In that case we know that $2s_{\text{gap}}^2=(s_{\text{gap}})^2$, so Eq. (2) can be used to calculate the mean velocity and the flow in the system. For the flow we thus get the asymptotic form:

$$q=\frac{(1-\rho)^2}{\sigma\rho} \quad \text{for} \quad \rho\rightarrow 1. \tag{3}$$

In Fig. 2 it can be seen that this is in excellent agreement with the results of the numerical calculations.

The above mentioned increase of the maximum flow can be traced back to differences in the statistics of the free flow. Note that when performing the continuous limit we generalized the random deceleration by zero or 1 in the CA model to a deceleration that is continuously equipartitioned between zero and 1 (this corresponds to setting the parameters to $a_{max}=1$ and $\sigma=1$). In this way the CA model and the corresponding continuous model assume the same values for the mean acceleration and the mean velocity in the free flow.

Other statistical parameters, however, are still different. The most important parameters are the variance of the velocity and of the acceleration in the free flow, because they determine the probability of interactions between the cars. In the continuous model used so far these are much lower than in the CA. This reduces the number of interactions and the probability that interacting cars cause a jam. Details on the role that the variance plays in determining interaction parameters are discussed in the next section.

To show that an adjustment of the continuous model to the CA is possible in the maximum flow regime we choose the parameters v_{max}, a_{max}, and σ in such a way that the mean velocity $\langle v\rangle$, the mean acceleration $\langle a\rangle$, and the acceleration variance $\langle a^2\rangle-\langle a\rangle^2$ in the free flow are equal in both models.

Choosing the parameters in this way yields a slightly higher velocity variance in the continuous model because the conditions require $\sigma > a_{max}$, so some cars are decelerated to a velocity that does not allow them to reach v_{max} within one time step again. An adjustment of the velocity variance would require the usage of a more complicated distribution in the randomization step.

Now the discrete and the continuous model compare very favorably in the free flow and the maximum flow regime, while the behavior is, as expected, still completely different for high densities (see Fig. 2).

The important result we have to keep in mind is that the maximum flow is determined by the free flow statistics. This fact may allow greatly simplified phenomenological theories of the CA dynamics.

IV. FREE FLOW STATISTICS

We now try to get a rough quantitative idea of how the statistics of the free flow determine the maximum flow. For this end we look at two neighboring cars that are assumed to move freely without any interactions. It will be possible to see which parameters determine the probability of interactions.

If the cars do not interact with each other, their individual velocity distributions $P_{v_1}(v_1,t)$ and $P_{v_2}(v_2,t)$ are uncorrelated and the time evolution of the distribution $P_g(x,t)$ of the gap between the two of them can be described by the simple master equation

$$P_g(x,t+1)=\int P_g(x+v_1-v_2,t)P_{v_1}(v_1,t) \times P_{v_2}(v_2,t)dv_1dv_2. \tag{4}$$

Taking moments of the gap distribution we find

$$\begin{aligned}\langle x\rangle(t+1)&=\langle x\rangle(t)+(\langle v_2\rangle-\langle v_1\rangle),\\ \delta x^2(t+1)&=\delta x^2(t)+\delta v_1^2(t)+\delta v_2^2(t),\end{aligned} \tag{5}$$

where δy^2 denotes the variance of the quantity y.

Now two simple cases will be looked at: The first case is that the last interaction of the cars has taken place a long time ago, so the velocity distributions of both cars are equal and stationary. The second case is that the cars are in the phase of acceleration, so the distributions are not necessarily equal and are certainly time dependent.

In the first case the mean gap between the cars is constant, while the width of the gap distribution increases as $\sqrt{t}$. Having two cars with a mean gap $\langle x\rangle$ we can estimate the mean time between interactions as the time it takes for the width of the distribution to reach a threshold where interaction takes place, say $\langle x\rangle - v_{\max}$. Averaging over all cars we get the mean time τ between interactions:

$$\tau \approx \alpha \frac{\left(\frac{1}{\rho}-1-v_{\max}\right)^2}{\langle v^2\rangle-\langle v\rangle^2}, \tag{6}$$

where α is a parameter of order 1 and ρ denotes the density of cars in the system.

From this expression we see that the number of interactions in the homogeneous flow is proportional to the velocity variance. On the other hand, inserting the value for the mean gap yields a value of, for instance, $\tau \approx 20$ in the region of maximum flow. This means that even though we know that interactions are of great importance in this regime they are still rare in a homogeneous flow. In addition it can be shown that interactions between at least three cars are needed to decelerate the cars to a velocity close to zero. Such events are extremely rare, as can be seen in Fig. 3. This already gives us a hint that the decrease of the flow cannot appear due to interactions in the homogeneous flow but must be a consequence of the appearance of jams.

Next we look at two accelerating cars. If the acceleration distribution is denoted by P_a, the master equation for the velocity distribution reads:

$$P_v(v,t+1)=\int P_v(v-a,t)P_a(a)da, \tag{7}$$

from which we get

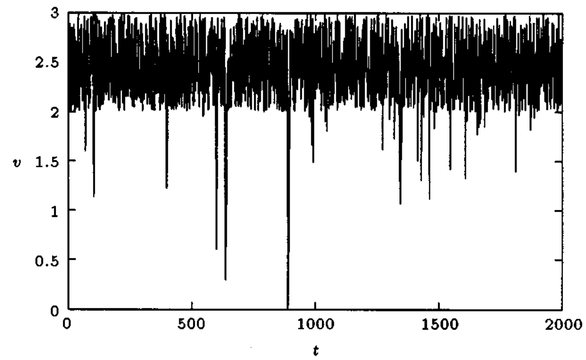

FIG. 3. Time evolution of the velocity of a randomly chosen car slightly below the density of maximum flow ($\rho = 0.17$).

$$\begin{aligned}\langle v\rangle(t+1)&=\langle v\rangle(t)+\langle a\rangle,\\ \delta v^2(t+1)&=\delta v^2(t)+\delta a^2.\end{aligned} \tag{8}$$

If we assume that the two cars start from $v=n_{\text{gap}}=0$ and the successor starts accelerating one time step after his predecessor $[P_{v_1}(t)=P_{v_2}(t-1)]$, we get for the evolution of the gap:

$$\begin{aligned}\langle x\rangle(t)&=\langle a\rangle t,\\ \delta x^2(t)&=(2t+1)\,\delta a^2.\end{aligned} \tag{9}$$

So the probability that the cars interact again during the phase of acceleration is determined by the ratio

$$r_a=\frac{\langle a\rangle^2}{\delta a^2}. \tag{10}$$

This parameter is in fact the most important parameter in the maximum flow region. This can be seen if we perform the following limit in the continuous model:

$$\sigma \to 0, \quad \langle a\rangle \to 0, \quad \text{with } \frac{\sigma}{\langle a\rangle}=2. \tag{11}$$

In Fig. 4 the fundamental diagram for this limit is shown. To calculate the limit a sequence of models with decreasing but nonzero σ was considered. We see that the amount by which the maximum flow changes when the limit is performed is comparatively small, because r_a is kept contant. Note that this limit is different from the deterministic limit

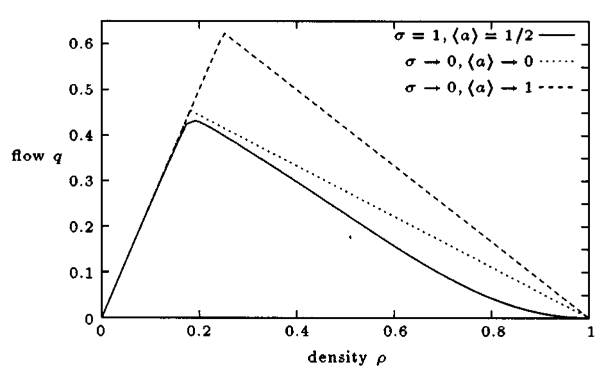

FIG. 4. The limit $\sigma \to 0, \langle a\rangle \to 0, \sigma/\langle a\rangle = 2$.

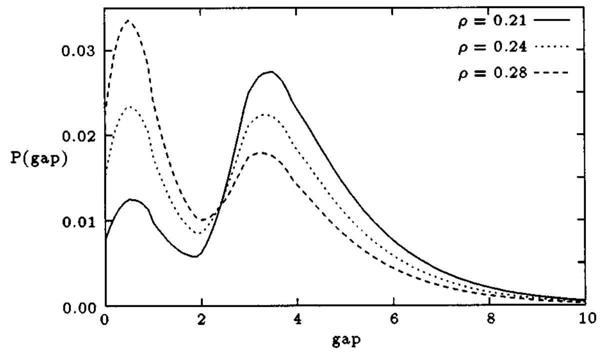

FIG. 5. Probability distribution of distances for different densities. Parameters chosen are $v_{\max}=3$, $a_{\max}=1$, $\sigma=1$.

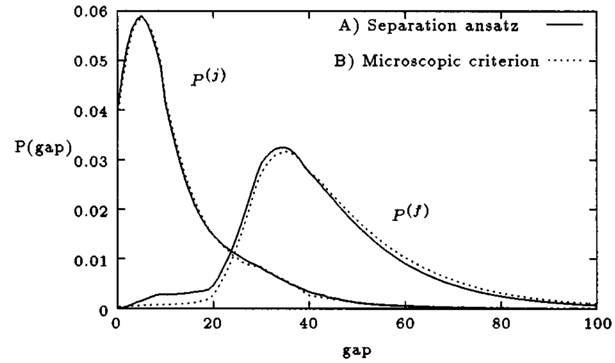

FIG. 6. (a) Probability distributions for the free flow and the congested flow phases, respectively, obtained from the separation ansatz. (b) Probability distributions obtained from a microsimulation compared with the distributions shown in (a). The microscopic criterion used to construct the distributions is $v \geqslant v_{\max}/2$ (free flow).

$(\sigma/\langle a\rangle \to 0)$, which is also depicted in Fig. 4 , in spite of the fact that there is no more noise in the free flow.

V. PHASE SEPARATION

Looking at the fundamental diagram we see that the flow density relation can be remarkably well approximated by a linear function over a wide range of densities in the congested flow region. The most obvious explanation for this is that the system decomposes into two phases, a phase of free flow and a phase of congested flow. The two phases are in a kind of dynamical equilibrium with each other, which closely resembles the thermal equilibrium between a liquid and a vapor in their coexistence regime. This picture will be exploited subsequently, but note that we have a system far from equilibrium here without any thermodynamic potential. Because of the equilibrium between the congested flow phase and the free flow phase any changes in the overall density of the system simply result in changes of the fraction of cars that can be found in either of the two phases, whereas the properties of the phases (i.e., mean flow and mean density) remain unchanged. To support this very suggestive picture we look at the gap distribution for different densities.

Figure 5(a) shows the gap distribution of the continuous model for different overall densities. Each of the distributions clearly exhibits two maxima. The positions of the maxima do not change significantly over a wide range of densities. Note that the lower maximum of the distribution is assumed at a nonzero value. This means that the equilibrium state of congestion in the two-phase region is not that of a ‘‘densely packed’’ queue of cars but rather a state with intermediate density.

This contrasts with the behavior of the original cellular automaton. In the CA model the maximum of the gap distribution of the congested regime is assumed for the value zero, which means that the jam is densely packed. This behavior is clearly unrealistic. In reality the spontaneously formed jams have a density significantly lower than for instance the density of a jam building up behind a blockage.

The qualitative picture described above can also be quantified. For this end we look at a series of N simulations in the range of densities $\rho_k (k=1,\dots,N)$ where we expect phase separation to take place. For these simulations we get gap distributions $P_k(x)$. Now assume that the $P_k(x)$ can be written in the form

$$P_k(x)=n_k P^{(j)}(x)+(1-n_k)P^{(f)}(x), \tag{12}$$

where $P^{(j)}$ and $P^{(f)}$ are the gap distributions of the jammed regions and the free regions, respectively. We assume that $P^{(j)}$ and $P^{(f)}$ are independent of the density ρ_k. The n_k are not independent variables, but are instead determined by the fact that the sum of all gaps has to be equal to the length of the system minus the space occupied by the cars. The corresponding equation for the n_k is derived multiplying Eq. (12) by x and integrating over the whole range of possible gaps. The integration yields

$$n_k=\frac{\langle x\rangle_f-\langle x\rangle_k}{\langle x\rangle_f-\langle x\rangle_j}, \tag{13}$$

where $\langle x\rangle_f$, $\langle x\rangle_j$, and $\langle x\rangle_k$ denote the mean gap with respect to the distributions $P^{(f)}$, $P^{(j)}$, and P_k. Equations (12) and (13) can now be used to compute the unknown distributions P_j and P_f. Note that Eq. (12) is basically a system of nonlinear equations for the $2N_b$ unknown $P^{(f)}(x_i)_{i=1,\dots,N_b}$, $P^{(j)}(x_i)_{i=1,\dots,N_b}$, where N_b is the number of bins used to describe the distributions. In order to solve this system of equations, one has to provide at least two data sets for different $P_k(x)$. If more data sets are provided, there will not be an exact solution to the problem any more and only the best approximate solution with respect to some appropriately defined distance measure can be found. This distance measure will be defined below.

It is also possible to decompose the gap distribution into a distribution for a jammed state and a free state using certain microscopic information about the individual events instead of a great number of different experiments. A simple guess would be to attribute cars to the jammed state if their velocity is below some threshold, say $v_{\max}/2$, and to the free state if it is above the threshold. The very details of the criterion used to separate the distributions do not seem to be crucial; many different criteria work with sufficient accuracy.

Figure 6 shows the gap distributions arising from the two decomposition methods. We see that they agree reasonably except for the small gaps in the free phase. The microscopic method attributes less cars to the free phase in this region, which is not surprising, because a simple threshold criterion is not able to distinguish between slow cars in a jam and slow cars in the free phase.

The distributions $P^{(f)}$ and $P^{(j)}$ are worth a closer look. First, what may be a little unexpected is the fact that the distribution of the free phase $P^{(f)}$ assumes nonvanishing values down to gaps of size zero. In this sense the ''free'' phase is different from really free conditions at very low densities, where there is a sharp cutoff at nonvanishing values for the gap size. The small gaps originate from interactions between the cars that do not suffice to finally cause a jam. Clearly the number of such events in the system is proportional to the number of cars in the free phase as long as the density in the free phase does not change significantly. So it is not surprising that the ansatz (12) brings out these events automatically. The time evolution of the velocity of a randomly chosen car in Fig. 3 also shows events of this kind. Again the picture of a liquid and its vapor may be helpful: The interactions that do not cause macroscopic jams are an analog to randomly generated aggregations of gas molecules that do not reach the critical size allowing them to evolve into a macroscopic droplet.

Next we look at the gap distribution $P^{(j)}$ for the jammed state. The distribution exhibits a peculiar behavior near gaps of size $v_{\max}$, where more events are counted than expected. The additional events counted in this range originate from the outflow region of the jams. As the number of such events is proportional to the number of jammed cars as long as the length distribution of the jams does not change too much, the ansatz (12) therefore attributes them to the jammed phase. The assumption of a constant length distribution of the jams is justified, because direct interactions between the jams are rare.

One important thing we have to keep in mind is the fact that different phases (i.e., spontaneously formed macroscopic inhomogeneties) can only exist in a range of densities $\rho_{c_1}<\rho<\rho_{c_2}$, where ρ_{c_1} and ρ_{c_2} are the densities in the free and the jammed phases, respectively.

After having understood how phase separation takes place in the continuous model we can compare this to the cellular automaton. We can again use the same ''decomposition ansatz'' (12) and find the discrete distributions $P^{(j)}$ and $P^{(f)}$ that fit the ansatz best. Defining an appropriate distance measure we can then compute the distance between the vector space spanned by the ansatz (12) and the simulation results.

A reasonable measure for the distance between two binned distributions q_i and p_i is

$$d(q,p)=\frac{\Sigma_i(q_i-p_i)^2}{\Sigma_i q_i p_i}. \tag{14}$$

The denominator is needed in this case to make sure that $d(q,p)\approx d(\hat{q},\hat{p})$ if (q,p) and $(\hat{q},\hat{p})$ are binned distributions with different resolutions corresponding to the same underlying continuous distribution. Otherwise the continuous model and the cellular automaton could not be compared directly.

To compare the continuous and the discrete models the decomposition was performed using experiments with densities ranging in an interval of $\Delta\rho=0.1$. Then the mean distance between the space spanned by the ansatz and the experimental distributions was computed using the above distance measure. For the discrete model the distance was approximately four times as large as for the continuous model, the order of magnitude being 10^{-2}.

The result again is not too surprising. The distance is considerably smaller in the continuous case due to the fact that the system has more degrees of freedom, but the picture of the separating phases is justified very well in both cases.

VI. CONCLUSIONS

The dynamics of the Nagel-Schreckenberg model bears considerable structure. Many of the interesting features of this model, however, are masked by the limitations of its discrete state space. It has been shown that an analogous model with continuous state variables can be constructed as the limit of a series of discrete models with different spatial resolutions.

In the continuous model the dynamics of the system for different densities can be nicely resolved into the following stages.

$0<\rho\leqslant\rho_{c_1}$: Free flow. Interactions between the cars are rare, leading to the formation of small ''droplets'' which immediately dissolve. Each car travels approximately at its desired speed (physical analog: dilute gas).

$\rho_{c_1}<\rho\leqslant\rho_{c_2}$: Phase separation. The system decomposes into regions of ''free'' flow and jammed regions (physical analog: saturated vapor in equilibrium with liquid phase).

$\rho_{c_2}<\rho\leqslant 1$: Rehomogenization. The whole system is congested (physical analog: compressible liquid without coexisting vapor). Because of the exponential tail of the distance distribution it is possible that fast cars can exist, but such cars get immediately dissolved in the ''liquid.''

Spontaneously formed jams can only exist for densities ρ between ρ_{c_1} and ρ_{c_2}. This is in fact a well known feature of hydrodynamical models of traffic flow [1] and of microscopic models with more complicated, deterministic dynamics [4,5,13]. So far it was not clear whether or not there is an analog for that in the CA model. Note that the lower critical density ρ_{c_1} is not identical with the density of maximum flow, but slightly lower. It has been found [14] that the time the system needs to relax into an equilibrium state diverges at this density.

Using continuous variables also allows one to adjust different statistical parameters of the free flow and of the highly congested flow independently, whereas in the original model one parameter determines the complete statistics. This is a highly desirable feature when it comes to the calibration of the model to empirical data. First experiences with the calibration of the model [12] also seem to indicate that the behavior of the continous model for high densities is essential to reproduce spontaneous jamming and dense jamming behind a blockage, for instance, quantitatively within one model.

It has been shown that much of the dynamics of the model in the region of maximum flow can be understood looking only at the statistics of the free flow region. This may be a basis for simplified theories of the cellular automaton and related models. As a first simple example of this the important role of the acceleration noise could be elucidated looking at the statistics of two noninteracting cars.

The main differences between the CA model and the continuous model arise in the limit of very high density. Apart from that limit the dynamics in both models are not substantially different. So despite the great advantages that the continuous model bears, its investigation has confirmed the validity of the calculations performed within the framework of the original CA model.

APPENDIX: THE INTERMEDIATE MODELS

The intermediate models used for the fundamental diagrams in Fig. 1 are defined as follows:

$$v_{des}=\min[v(t)+a_{max},v_{max},n_{gap}(t)],$$
$$v(t+1)=\max[0,v_{des}-n_{ran,0,a_{max}}], \qquad (15)$$
$$x(t+1)=x(t)+v(t+1),$$

where all variables are integers. For the results displayed in Fig. 1 v_{max} was chosen to be an integer multiple of 3. The maximum acceleration a_{max} was set to $v_{max}/3$, $n_{ran,0,a_{max}}$ denotes an integer random number between 0 and a_{max}. The gap is calculated as the difference of the positions that the centers of the cars have minus the length of one car, which is equal to $1/\rho_{max}$.

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